ONE-DIMENSIONAL QUASI-STATICAL MOTIONS OF SOIL

(ODNOMERNYE KVAZISTATICHESKIE DVIZHENIIA GRUNTA)

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References [1,2] suggest a model for description of soil motions and similar media. For determination of detailed properties of such a model and development of qualitative properties of motions described by the model, it is necessary to consider simple formulations of problems which, while allowing a complete analysis under most general assumptions about the model characteristic functions, permit one to study the general qualitative properties of the motions for these simple formulations without concretely defining the functions considered. Among such functions is the function f from the relation $p = f(\rho, \rho_*)$ describing volumetric deformation of the medium and the function F from the plasticity condition $J_2 = F(p)$ characterizing shear deformation.

This work carries out a general investigation of one such simply formulated problem, namely, the one-dimensional motion of a medium under slowly varying externally applied loadings when it is permissible to neglect accelerations in the equations of motion. Such motions are termed quasi-statical. The following qualitative properties of the problem have been established resulting from analysis of problem solution. First, consideration of the plane one-dimensional problem leads to the conclusion that with the increase of the axial compressive stress $-\sigma_x$, after the elastic shear deformation is exceeded and further deformation is plastic, there will be an instant when in the presence of sufficiently large stresses the shear will again occur elastically. If in addition one makes a natural assumption that the shear elastic region $F(p)$ remains bounded for $p \to \infty$ (of the order of G, the shear modulus) then with the increase in compressive stresses there will occur (now under elastic shear conditions) a drawing together of the axial and lateral stresses σ_x and σ_y. The state of stress will approach a state of hydrostatic compression. It is worth noting in this connection that such an effect, apparently, actually takes place under the conditions of natural stratifications in rocks: there exists an opinion that at great depth the lateral rock pressure is close to the vertical pressure.

Further, the study of the centrally symmetrical problem leads to the conclusion that if the function $F(p)$ is such that $\sqrt{(3F(p))} - F'(p) < 0$ for certain values of p, then in the region of motion there occur limit lines, similar to the analogous problem in gasdynamics (stationary gas flow of the source or sink type). In the given situation the occurrence of the limit lines is not permissible on physical grounds. Therefore one should accept the assumption that for any real media the function $F(p)$ should satisfy the condition $\sqrt{(3F(p))} > F'(p)$ for all values of p (analogous to gasdynamics, for example where it is necessary to assume that for all gases the adiabatic characteristic $\gamma > 1$).

Finally, the consideration of the centrally symmetrical case shows that if in the infinite space, filled by a medium, there is a spherical cavity in which the pressure is slowly increasing, then the equilibrium is possible with the finite radius of the cavity only for the pressures not exceeding some limit value. With pressure approaching the limit value the radius of the cavity tends to infinity. This indicates that the medium cannot sustain arbitrarily large values of pressure within the cavity; with increase in pressure the resistance of the medium decreases. Its characteristics tend to approach those of a liquid. If the initial radius of the cavity is zero, then for pressures less than the limit values the cavity is not formed at all. In approaching the limit pressures a cavity of an arbitrary radius is formed, while for pressures above the limit an equilibrium is not possible (in any case a symmetric form of equilibrium). These results are somewhat unexpected and their prediction would have been difficult.

1. We shall consider the motion in the Cartesian, cylindrical and spherical system of coordinates corresponding to the motions with the plane, cylindrical and central symmetry. In these three cases we shall have the following expressions for the components of stress tensors and deformation velocities (x is everywhere a linear coordinate):

for $\nu = 0$

$$\sigma_{xx} = \sigma, \qquad \sigma_{yy} = \sigma_{zz} = \sigma_1, \qquad 3p = -(\sigma + 2\sigma_1)$$
$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = e_{zz} = 0$$

for $\nu = 1$

$$\sigma_{xx} = \sigma, \quad \sigma_{\theta\theta} = \sigma_1, \quad \sigma_{zz} = \sigma_2, \quad 3p = -(\sigma_1 + \sigma_2 + \sigma_3)$$
$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{\theta\theta} = \frac{u}{x}, \qquad e_{zz} = 0 \tag{1.1}$$

for $\nu = 2$

$$\sigma_{xx} = \sigma, \qquad \sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \sigma_2, \qquad 3p = -(\sigma + 2\sigma_1)$$
$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{\theta\theta} = e_{\varphi\varphi} = u / x$$

Here u is the only nonzero component of the velocity vector.

One-dimensional motion is described by the following system of relationships [2]:

$$\begin{gathered}
\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} + \nu\frac{u\rho}{x} = 0, \qquad \rho\frac{du}{dt} = \frac{\partial\sigma}{\partial x} + \nu\frac{\sigma - \sigma_1}{x} \\
\frac{d(\sigma + p)}{dt} + \lambda(\sigma + p) = 2G\left[\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \nu\frac{u}{x}\right)\right] \\
\frac{d(\sigma_1 + p)}{dt} + \lambda(\sigma_1 + p) = 2G\left[\delta\frac{u}{x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \nu\frac{u}{x}\right)\right] \\
\frac{d(\sigma_2 + p)}{dt} + \lambda(\sigma_2 + p) = 2G\left[0 - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \nu\frac{u}{x}\right)\right] \\
p = f(\rho, \rho_*)\,e(\rho_* - \rho)\,e(\rho - \rho_0) \equiv f^\circ(\rho, \rho_*), \quad \frac{d\rho_*}{dt} = \frac{d\rho}{dt}\,e(\rho - \rho_*)\,e\left(\frac{d\rho}{dt}\right)
\end{gathered} \tag{1.2}$$

where $J_2 < F(p)$ and

$$J_2 \equiv \tfrac{1}{2}\left[(\sigma + p)^2 + (\sigma_1 + p)^2 + (\sigma_2 + p)^2\right] \quad \text{for} \quad \nu = 1$$
$$J_2 \equiv \tfrac{1}{2}\left[(\sigma + p)^2 + 2(\sigma_1 + p)^2\right] \quad \text{for} \quad \nu \neq 1 \tag{1.3}$$

In these formulas

$$\begin{gathered}
\lambda = \frac{2GW - F'(p)\,dp/dt}{2F(p)}\,e\left[J_2 - F(p)\right]e\left[2GW - F'(p)\frac{dp}{dt}\right] \\
2GW \equiv 2G(\sigma + p)\left(\frac{\partial u}{\partial x} - \delta\frac{u}{x}\right) \quad \text{for} \quad \nu \neq 1 \\
2GW \equiv 2G\left[(\sigma + p)\frac{\partial u}{\partial x} + (\sigma_1 + p)\frac{u}{x}\right] \quad \text{for} \quad \nu = 1 \\
e(a) = \begin{cases} 1, & a \geqslant 0, \\ 0, & a < 0, \end{cases} \qquad \delta = \begin{cases} 0, & \nu = 0 \\ 1, & \nu = 1,2 \end{cases}
\end{gathered} \tag{1.4}$$

If the shear occurs elastically, i.e. $J_2 < F(p)$, then $\lambda \equiv 0$, if not, then $\lambda > 0$.

Let us pass to the Lagrangian variables by using

$$x = x(a, t), \qquad u = \frac{\partial x}{\partial t} = x_t, \qquad \frac{\partial}{\partial x} = \frac{1}{x_a}\frac{\partial}{\partial a} \tag{1.5}$$

From (1.2) we have ((1.4) is transformed analogously)

$$\frac{\partial}{\partial t}(\rho x_a x^\nu) = 0, \qquad \rho x_a x_{tt} = \sigma_a + \nu\frac{\sigma-\sigma_1}{x}x_a \tag{1.6}$$
$$\frac{\partial}{\partial t}(\sigma + p) + \lambda(\sigma + p) = \frac{4G}{3}\Big(\frac{\partial \ln x_a}{\partial t} - \frac{\nu}{2}\frac{\partial \ln x}{\partial t}\Big)$$
$$\frac{\partial}{\partial t}(\sigma_1 + p) + \lambda(\sigma_1 + p) = -\frac{2G}{3}\Big[\frac{\partial \ln x_a}{\partial t} - (3\delta - \nu)\frac{\partial \ln x}{\partial t}\Big]$$

Substituting

$$x^{\nu+1} = y, \qquad a^{\nu+1} = z \tag{1.7}$$

we obtain from (1.6)

$$\rho y_z = \varphi(z)$$
$$\frac{\rho y_z}{(\nu+1)^2}\, y^{-\frac{2\nu}{\nu+1}}\Big(y_{tt} - \frac{\nu}{\nu+1}\frac{y_t^2}{y}\Big) = \sigma_z + \frac{\nu}{\nu+1}\frac{\sigma-\sigma_1}{y}y_z \tag{1.8}$$
$$\frac{\partial}{\partial t}(\sigma + p) + \lambda(\sigma + p) = \frac{4G}{3}\frac{\partial}{\partial t}\Big[\ln y_z - \frac{3\nu}{2(\nu+1)}\ln y\Big]$$
$$\frac{\partial}{\partial t}(\sigma_1 + p) + \lambda(\sigma_1 + p) = -\frac{2G}{3}\frac{\partial}{\partial t}\Big[\ln y_z - \frac{3\delta}{\nu+1}\ln y\Big]$$

Considering further the motion to be quasi-stationary and originating from the state of constant density ρ_{00}, we finally obtain the system (1.8), with the first two equations of the form

$$\rho y_z = \rho_{00}, \qquad \sigma_z + \frac{\nu}{\nu+1}\frac{\sigma-\sigma_1}{y}y_z = 0 \tag{1.9}$$

2. Let us consider the plane case ($\nu = 0$). If shear occurs elastically, i.e. $\lambda = 0$, then from (1.8), (1.9), considering that $G = G(\rho)$ and for $t = 0$, $\rho = \rho_{00}$, $\sigma + p = 0$, we obtain

$$\sigma = -f^\circ(\rho, \rho_*) - \frac{4}{3}\int_{\rho_{00}}^{\rho}\frac{G(\rho)}{\rho}d\rho, \qquad \sigma_1 = -f^\circ(\rho, \rho_*) + \frac{2}{3}\int_{\rho_{00}}^{\rho}\frac{G(\rho)}{\rho}d\rho, \tag{2.1}$$
$$\rho = \rho(t), \qquad y = \frac{\rho_{00}}{\rho}z + y_1(t)$$

where $\rho(t)$ and $y_1(t)$ are arbitrary functions to be defined by two boundary conditions. It may happen that before shear becomes plastic ρ will attain a value ρ_*, and ρ_* will begin to vary. This variation may be easily determined from the ρ_*-equation in (1.2). Plastic shear will occur when the following condition becomes satisfied:

$$\int_{\rho_{00}}^{\rho}\frac{G}{\rho}d\rho = \pm\sqrt{\tfrac{3}{4}F[f^\circ(\rho, \rho_*)]} \tag{2.2}$$

Here the upper limit corresponds to the compression of the medium, i.e. the condition $\sigma < \sigma_1$, and the lower limit to rarefaction, i.e. the condition $\sigma > \sigma_1$.

If shear is plastic then $3/4(\sigma + p)^2 = F(p)$ and the solution is obtainable by simple substitution of the first two formulas in (2.1) by

$$\sigma = -f^\circ(\rho, \rho_*) \mp \tfrac{2}{3}\sqrt{3F[f^\circ(\rho, \rho_*)]}$$
$$\sigma_1 = -f^\circ(\rho, \rho_*) \pm \tfrac{1}{3}\sqrt{3F[f^\circ(\rho, \rho_*)]} \qquad (2.3)$$

Consider the expression for λ. When $J_2 = F(p)$ the value of λ is non-zero if $2GW - F'(p)dp/dt > 0$. In this case

$$2GW - F'(p)\frac{dp}{dt} = 2G(\sigma + p)\left(-\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right) - \qquad (2.4)$$
$$-F'(p)\left[\frac{\partial f^\circ}{\partial\rho} + \frac{\partial f^\circ}{\partial\rho_*}e(\rho-\rho_*)e\left(\frac{\partial\rho}{\partial t}\right)\right]\frac{\partial\rho}{\partial t}$$
$$= -\frac{\partial\rho}{\partial t}\left\{F'(f^\circ)\left[\frac{\partial f^\circ}{\partial\rho} + \frac{\partial f^\circ}{\partial\rho_*}e(\rho-\rho_*)e\left(\frac{\partial\rho}{\partial t}\right)\right] \mp \frac{4}{3}\frac{G}{\rho}\sqrt{3F(f^\circ)}\right\}$$

This expression shows that if plastic shear began during rarefaction, i.e. when $\partial\rho/\partial t < 0$, then it will persist to the end of rarefaction up to the point of loosening ($\rho = \rho_0$) since (2.4) is positive (lower sign). If, however, the shear began during compression (upper sign) then, depending on the function f° and F, it may persist indefinitely, since there will occur such a time when λ becomes zero and thereafter shear will occur elastically. The latter will take place particularly if p increases without limit with the increase in ρ, but $F(p)$ remains bounded. In this case the solution should be continued with the aid of the equations describing the elastic shear. It is easily verified that in this case $J_2 < F(p)$.

3. Let us study the spherical case. We will consider monotonic processes of loading and unloading from the initial homogeneous condition. In this case one may assume that p and ρ are well defined by some relationship $p = \phi(\rho)$ (see (1.2)). Initially shear will occur elastically, i.e. $\lambda = 0$ and $J_2 < F(p)$. At this stage, integrating the corresponding equations in (1.8) and assuming for simplicity that $G = \text{const}$, we obtain

$$\sigma + p = \tfrac{4}{3}G\left(\ln y_z - \ln\frac{y}{z}\right), \qquad \sigma = -\varphi\left(\frac{\rho_{00}}{y_z}\right) + \tfrac{4}{3}G\ln\frac{y_z z}{y} \qquad (3.1)$$
$$\sigma_1 = -\varphi\left(\frac{\rho_{00}}{y_z}\right) - \tfrac{2}{3}G\ln\frac{y_z z}{y}, \qquad y_z = \frac{\rho_{00}}{\rho}$$

For determination of $y(z)$ we obtain further the equation

$$y_{zz} = \psi(y_z)\,\frac{y_z}{z}\left[\frac{zy_z}{y}\left(1 - \ln\frac{zy_z}{y}\right) - 1\right] \tag{3.2}$$

$$\psi(y_z) = \left[1 + \frac{3\rho_{00}\varphi'(\rho_{00}/y_z)}{4Gy_z}\right]^{-1} = \left(1 + \frac{3\varphi'\rho}{4G}\right)^{-1} \leqslant 1 \tag{3.3}$$

Solving from this for $y(z)$ under corresponding boundary conditions we will obtain a complete solution for the elastic shear problem.

Let us investigate this equation. Since $\psi(y_z) > 0$, $y_z > 0$ and the expression in the brackets of Equation (3.2) is negative for all values of zy_z/y on which it depends only (except one value $zy_z/y = 1$), we conclude that

$$y_{zz} \leqslant 0, \text{ i.e. } \frac{d}{dz}\frac{1}{\rho} \leqslant 0 \tag{3.4}$$

This indicates that at the stage when shear occurs elastically density is a nondiminishing function of the distance from the center.

This result appears somewhat paradoxical in the case of deformation due to a pressure rise in a certain spherical cavity in the medium when, it would appear, the medium should condense. The result derived is connected with the law of elasticity assumed in the model considered. However, since an elastic response takes place for sufficiently small deformations, as we shall see later, for which the density variations are generally unimportant and for which, neglecting small quantities, one may obtain a solution by way of the solution for the theory of elasticity problem, the result obtained should not be too disturbing.

Equation (3.2) permits extension along y and z an equal number of times, therefore a lowering of order is possible. It has a particular solution $y = cz$. At the same time $d\rho/dz = 0$, $zy_z/y = 1$, the above noted root of the right-hand side of (3.2), $\sigma = \sigma_1 = -\phi(\rho) = $ const. This solution corresponding to hydrostatic compression (rarefaction) is realized under specially given boundary conditions (similar deformation).

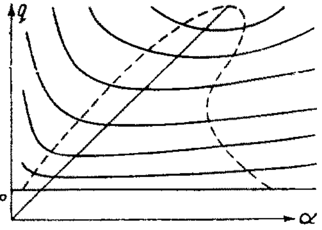

Fig. 1.

For investigation of the remaining solutions of (3.2) introduce variables

$$y_z = q, \qquad y/z = \alpha \tag{3.5}$$

Then (3.2) is transformed into an equation of first order

$$q_\alpha = \psi(q)\,\frac{q}{\alpha}\left[1 - \frac{(q/\alpha)\ln(q/\alpha)}{(q/\alpha) - 1}\right] \equiv \psi(q)\,h\left(\frac{q}{\alpha}\right) \tag{3.6}$$

It is easily shown that $h(v) > 0$ for $0 < v < 1$ and $h(v) < 0$ for $1 < v$, and $h(1) = 0$. Therefore the field of integral curves of (3.6) is as shown in Fig. 1. If for unlimited increase in pressure p the density remains limited (bounded), then the field of the integral curves will be limited from below by a limit straight line, i.e. solution $q = q_\infty = \rho_{00}/\rho_\infty = \text{const}$, where ρ_∞ is the density at $p = \infty$, and in the opposite case by the axis $q = 0$. At the left it is bounded by the solution $\alpha = 0$.

If the relation $q = q(\alpha)$ is known, then all the other characteristics are found from the formulas

$$\rho = \rho_{00}/q, \qquad p = \varphi(\rho_{00}/q)$$

$$\sigma = -\varphi(\rho_{00}/q) + \tfrac{4}{3}G\ln\frac{q}{\alpha}, \quad \sigma_1 = -\varphi(\rho_{00}/q) - \tfrac{2}{3}G\ln\frac{q}{\alpha} \tag{3.7}$$

$$\ln z = \int\frac{d\alpha}{q-\alpha}, \qquad y = z\alpha$$

The qualitative nature of the relationships in (3.7) is shown in Fig. 2. The distribution of the quantities q, ρ, p, σ, along α is nonmonotonic. For $q = \alpha$ the first three quantities possess an extremum while the quantity σ is a strictly diminishing function of α. The latter stems from the formula

$$\frac{d\sigma}{d\alpha} = -4G\,\frac{\alpha\ln(q/\alpha)}{q/\alpha - 1} < 0 \tag{3.8}$$

following from (3.7) and (3.6). The distribution character for $\sigma_1(\alpha)$ is established in an analogous manner.

The solutions for elastic shear investigated above are valid only for $J_2 < F(p)$. The condition $J_2 = F(p)$ on the surface q, α will determine the boundary for the region of elastic shear. It can easily be shown that this condition yields the following equations for the boundary:

$$\alpha_1 = q\exp\left\{-\frac{1}{2G}\sqrt{3F[\varphi(\rho_{00}/q)]}\right\} \tag{3.9}$$

$$\alpha_2 = q\exp\left\{\frac{1}{2G}\sqrt{3F[\varphi(\rho_{00}/q)]}\right\} \tag{3.10}$$

so that the region of elastic shear is defined by the inequalities

$$\alpha_1(q) \leqslant \alpha \leqslant \alpha_2(q)$$

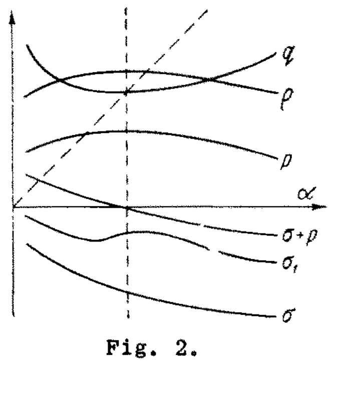

Fig. 2.

We will consider that $F(0) = 0$ and that the loosening occurs when $p = 0$, $\rho = \rho_0$. Then the lines (3.9), (3.10) on the plane q, α will meet at the point $q = q_0 = \alpha_0 = \rho_{00}/\rho_0$. In decreasing q from q_0 to q_∞ the line (3.9) is monotonically lowered into the point $q = q_\infty$, $\alpha = 0$ if $F(p) \to \infty$ for $p \to \infty$, or merges with the straight line $q = q_\infty$ for nonzero $\alpha = \alpha_1(q_\infty) < q_\infty$ (Fig. 1, dotted line) if $F(p)$ is limited. Curve (3.10), generally speaking, is nonmonotonic. It will approach the straight line $q = q_\infty$ asymptotically if $F(p) \to \infty$ for $p \to \infty$, or will merge with it for a finite $\alpha = \alpha_2(q_\infty) > q_\infty$ and limited $F(p)$. To the left of the curve (3.9) and to the right of (3.10) on the plane q, α, it is necessary to construct a solution with plastic shear. In approaching the line (3.9) from the region of elastic shear $\sigma + p \to 2/3 \sqrt{(3F(p))}$ (stretching), while in approaching (3.10) $\sigma + p \to -2/3 \sqrt{(3F(p))}$ (compression; see (3.7)). Transition to the plastic state occurs for a certain $z = z_*$, whereby the relations

$$y^e = y^p, \qquad \sigma^e = \sigma^p \tag{3.11}$$

expressing the continuity of deformations and radial normal stresses take place (e denotes a quantity from the elastic solution and p from the plastic one). Since in this case

$$J_2(\sigma^e, p^e) - F(p^e) \to 0, \qquad J_2(\sigma^p, p^p) - F(p^p) = 0$$

then for $z = z_*$ the relation $p^e = p^p$ will also be fulfilled, and consequently $\rho^e = \rho^p$, $y_z^e = y_z^p$. This means that

$$q^e = q^p, \qquad \alpha^e = \alpha^p \tag{3.12}$$

i.e. in the plane q, α the integral curves must pass continuously from the elastic into the plastic region.

4. For the state when shear is plastic, $J_2 = F(p)$ and we have

$$\sigma = -p \pm \tfrac{2}{3}\sqrt{3F(p)}, \qquad \sigma_1 = -p \mp \tfrac{1}{3}\sqrt{3F(p)} \tag{4.1}$$

Substituting (4.1) into (1.9) and taking into account the relation $p = \phi(\rho_{00}/y_z)$ we will obtain a basic equation for plastic shear

$$-\left[-1 \mp \frac{F'(p)}{\sqrt{3F(p)}}\right]\varphi' \frac{\rho_{00}}{y_z^2} y_{zz} \pm \frac{2}{3}\sqrt{3F(p)}\frac{y_z}{y} = 0 \tag{4.2}$$

This equation written in the form

$$-\left[1 \pm \frac{F'(p)}{\sqrt{3F(p)}}\right]\frac{dp}{dz} \pm \frac{2}{3}\sqrt{3F(p)}\frac{d\ln y}{dz} = 0$$

is integrated

$$y = \frac{A}{\sqrt{3F(p)}} \exp\left[\pm \frac{3}{2} \int \frac{dp}{\sqrt{3F(p)}}\right] \tag{4.3}$$

where $A = A(t)$ is a constant of integration. In principle, the dependence of y on z, containing two constants of integration, can be obtained by inverting the relation $p = \phi(\rho) = \phi(\rho_{00}/y_z)$, substituting in the result the inversion (4.3) and carrying out the quadrature. However, it is more convenient to obtain this dependence in a parametric form. Namely

$$\begin{gathered}\frac{dz}{dp} = \frac{dJ}{dp}\frac{1}{y_z} = \frac{\rho(p)}{\rho_{00}}\left\{\pm \frac{A}{2F(p)}\left[1 \mp \frac{F'(p)}{\sqrt{3F(p)}}\right] \exp\left[\pm \frac{3}{2}\int \frac{dp}{\sqrt{3F(p)}}\right]\right\} \\ z = \pm \frac{A(t)}{2\rho_{00}} \int \left\{\frac{\rho(p)}{F(p)}\left[1 \mp \frac{F'(p)}{\sqrt{3F(p)}}\right] \exp\left[\pm \frac{3}{2}\int \frac{dp}{\sqrt{3F(p)}}\right]\right\} dp + B(t)\end{gathered} \tag{4.4}$$

Formulas (4.3) and (4.4) yield the solution of Equation (4.2) with two arbitrary constants A and B; however, in coupling with the solution for the elastic shear it is convenient to use the variables q and α. Making this transformation we will obtain from (4.2) ((4.2) has no solution $y = cz$))

$$\frac{dq}{d\alpha} = \mp \frac{2}{3}\sqrt{3F}\,\frac{q^3}{\rho_{00}\varphi'(\rho_{00}/q)\,\alpha(q-\alpha)}\left[1 \mp \frac{F'}{\sqrt{3F}}\right]^{-1} \tag{4.5}$$

The integral curves of Equation (4.5) should continuously match on the lines (3.9), (3.10) with the integral curves of Equation (3.6), in accordance with the conditions (3.12). On the strength of (3.7) $\sigma + p > 0$ on the line (3.9), while on (3.10) $\sigma + p < 0$. Therefore, because of continuity in $\sigma + p$ on (3.9) and (3.10), it follows from Formulas (4.1) that in the plastic region to the left of (3.9) one should choose the upper sign in (4.1) and the subsequent formulas and to the right of (3.10) the lower sign. Equation (4.5) shows that to the right of (3.10) $dq/d\alpha < 0$ and to the left of (3.9) $dq/d\alpha < 0$ if $F(p)$ is such that

$$1 - F'(p)/\sqrt{3F(p)} > 0 \tag{4.6}$$

If the latter inequality can be violated, then to the left of (3.9) the integral curves (4.5) can have regions with positive slope.

Since at the point on an integral curve where the slope changes from negative to positive the sign of $d\alpha$ is changing (see (4.5)), then at that point the sign of dz will also change (see formula for z in (3.7)), i.e. at that point z will obtain a minimum value z_{min}. Beyond this point the solution becomes double-valued. A limit sphere $z = z_{min}$ corresponds to this point, the solution in the interior of which cannot be continued. A boundary (limit) line appears in the region where the solution is being constructed. The analogous phenomenon is well known in gasdynamics. In the case considered, this effect is tied to the properties of $F(p)$ for small p (in the region considered q increases, i.e. ρ and p decrease on the surface of a). For example, if $F(p) = (kp)^2$, which is equivalent to $\sigma_1 = c\sigma$, the condition for the appearance of the limit line will be $k < 1/2\sqrt{3}$, or $C > 0.25$. In future we will assume that the inequality (4.6) is satisfied and the limit line does not appear.*

Integral curves of Equation (4.5) are defined below the straight line $q = q_0 = \rho_{00}/\rho$, which corresponds to loosening of the medium ($p = 0$). Since q is monotonically decreasing with increase in a along the integral curves of Equation (4.5), while along the boundary of the elastic region (3.9) q increases, then each of the integral curves (4.5) intersects the boundary (3.9) at one point for $q > a$, i.e. each of the integral curves can be continued in only one way from the elastic into the plastic region for $q > a$, so that in further motion along the curve we shall remain in the plastic range up to the loosening condition. Analogously for $q > a$ and increasing q moving along any integral curve in the elastic region, we will reach the boundary (3.10), after which we pass continuously on a certain integral curve of Equation (4.5). In moving along (4.5) q will decrease, but since (3.10) has regions with negative slope it is not clear that we will not reach the boundary (3.10) again. In order to prove that this cannot happen for any integral curve of Equation (4.5), we will write down the general solution of this equation, dividing (4.3) by (4.4) and recalling that $y/3 = a(\rho_{00}/q)$; the result is

$$\alpha = \frac{\exp[\pm {}^3/_2\Phi_1(p)]}{\sqrt{3F(p)}}\left(\pm\frac{1}{2\rho_{00}}\Phi_{2\mp}(p) + \frac{B}{A}\right)^{-1} \tag{4.7}$$

Here

$$\Phi_1(p) = \int_{p_1}^{p} \frac{dp}{\sqrt{3F(p)}} \tag{4.8}$$

$$\Phi_{2\mp}(p) = \int_{p_1}^{p} \left\{\frac{\rho(p)}{F(p)}\left[1 \mp \frac{F'(p)}{\sqrt{3F(p)}}\right]\exp\left[\pm\frac{3}{2}\Phi_1(p)\right]\right\}dp \tag{4.9}$$

Here p and B/A are arbitrary. If a point is given on the boundary (3.10) (or (3.9)) through which the curve (4.7) should pass, then this

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* Note that the function $F(p)$ constructed as the result of experiments with sandy soils in [3] satisfies this condition.

yields the conditions for determination of p_1 and B/A. Indeed, conditions $a = a_{1,2}$, $q = q_1$ yield $p = \phi(\rho_{00}/q_1) = p_1$ which determines p_1 and $a_{1,2} = A/B \sqrt{(3F(p_1))}$; whence $B/A = 1/a_{1,2} \sqrt{(3F(p_1))}$. Substituting p_1 and B/A into (4.7) we obtain

$$\alpha = \frac{\Psi(p)\exp\left[\pm {}^3/_2\,\Phi_1(p)\right]}{\pm {}^1/_2\,\rho_{00}^{-1}\sqrt{3F(p_1)}\,\Phi_{2\mp}(p) + 1/\alpha_{1,2}} \qquad \left(\Psi(p) = \sqrt{\frac{F(p_1)}{F(p)}}\right) \tag{4.10}$$

Substituting here in place of $a_{1,2}$ Expressions (3.9) and (3.10)

$$\alpha_{1,2} = q_1 \exp\left[\mp\frac{1}{2G}\sqrt{3F(p_1)}\right] = \frac{\rho_{00}}{\rho_1}\exp\left[\mp\frac{1}{2G}\sqrt{3F(p_1)}\right]$$

we finally obtain

$$\dot{\alpha} = \frac{\rho_{00}\Psi(p)\exp\left[\pm {}^3/_2\,\Phi_1(p)\right]}{\pm {}^1/_2\sqrt{3F(p_1)}\,\Phi_{2\mp}(p) + \rho_1\exp\left[\pm {}^1/_2 G^{-1}\sqrt{3F(p_1)}\right]} \tag{4.11}$$

Choosing now the lower sign for the case $q < a$ we evaluate the integral in the denominator of (4.11), using the inequalities $\rho_\infty > \rho > \rho_1$ which are satisfied on the integral curve for $q < a$; as the result, using (4.9) and (4.8) we obtain

$$K \equiv -\frac{1}{2}\sqrt{3F(p_1)}\,\Phi_{2+}(p) <$$

$$< -\frac{\rho_1\sqrt{3F(p_1)}}{2}\int_{p_1}^{p}\left\{\frac{1}{F(p)}\left[1 + \frac{F'(p)}{\sqrt{3F(p)}}\right]\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\}dp$$

$$= \rho_1\sqrt{F(p_1)}\int_{p_1}^{p} d\left\{\frac{1}{\sqrt{F(p)}}\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\}$$

$$= -\rho_1\left\{1 - \sqrt{\frac{F(p_1)}{F(p)}}\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\}$$

Analogously, substituting ρ_∞ for $\rho(p)$ we obtain

$$K > -\rho_\infty\left\{1 - \sqrt{\frac{F(p_1)}{F(p)}}\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\}$$

Therefore the following inequality is satisfied for the denominator D of the right-hand side of (4.11):

$$\rho_\infty\left\{\frac{\rho_1}{\rho_\infty}\exp\left[-\frac{1}{2G}\sqrt{3F(p_1)}\right] - 1 + \sqrt{\frac{F(p_1)}{F(p)}}\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\} <$$

$$< D < \rho_1\left\{\exp\left[-\frac{1}{2G}\sqrt{3F(p_1)}\right] - 1 + \sqrt{\frac{F(p_1)}{F(p)}}\exp\left[-\frac{3}{2}\Phi_1(p)\right]\right\} \tag{4.12}$$

The p-dependent components on the right-hand side of (4.12) tend to zero for $p \to \infty$ in the case when $F(p)$ is bounded for $p \to \infty$ as well as when $F(p) \to \infty$ for $p \to \infty$; consequently, the right-hand side tends to

$$-\rho_1\left\{1-\exp\left[-\frac{1}{2G}\sqrt{3F(p_1)}\right]\right\}<0$$

for $p \to \infty$.

Therefore D becomes negative for sufficiently large p. This means that there exists a value of pressure $p^*(p_1 < p^* < \infty)$ for which $D = 0$, i.e. $a = \infty$. This indicates that the integral curves have horizontal asymptotes $q_* = q(p^*) < q_1$ for $a < q$.

We will show now that the integral curve, having its origin at the point on the boundary (3.10), does not cross this boundary anywhere else. For $q_* \leqslant q < q_1$ and using (4.12) we obtain from (4.11)

$$\alpha > \frac{\rho_{00}\Psi(p)\exp\left[-{}^3/_2\,\Phi_1(p)\right]}{\rho_1\{\exp\left[-{}^1/_2G^{-1}\sqrt{3F(p_1)}\right]-1+\Psi(p)\exp\left[-{}^3/_2\,\Phi_1(p)\right]\}}$$

On the other hand, we have along the boundary

$$\alpha=\alpha_2=q\exp\left[\frac{1}{2G}\sqrt{3F(p)}\right]<\frac{\rho_{00}}{\rho_1}\exp\left[\frac{1}{2G}\sqrt{3F(p)}\right]$$

Therefore in order to prove that $a > a_2$ it is sufficient to show that

$$\frac{\rho_{00}\Psi(p)\exp\left[-{}^3/_2\,\Phi_1(p)\right]}{\rho_1\{\exp\left[-{}^1/_2G^{-1}\sqrt{3F(p_1)}-1+\Psi(p)\exp\left[-{}^3/_2\,\Phi_1(p)\right]\right\}}>\frac{\rho_{00}}{\rho_1}\exp\frac{\sqrt{3F(p)}}{2G}$$

or

$$1-\exp\left[-\frac{1}{2G}\sqrt{3F(p_1)}\right]>\sqrt{\frac{F(p_1)}{F(p)}}\left\{1-\exp\left[\frac{-\sqrt{3F(p)}}{2G}\right]\right\}\exp\left[-{}^3/_2\,\Phi_1(p)\right]$$

Instead of the last inequality it is sufficient to show a stronger one

$$\frac{1-e^{-x_1}}{x_1}>\frac{1-e^{-x}}{x}\equiv\Phi(x),\quad x=\frac{1}{2G}\sqrt{3F(p)},\quad x_1=\frac{1}{2G}\sqrt{3F(p_1)}$$

for the condition $0 < x_1 < x$. But $\phi(x)$ is decreasing monotonically for $x > 0$, which is the proof required. The statement that the continuation of any integral curve from the elastic into the plastic region realized uniquely remains completely in the plastic region is proved by the same argument. Thus, the half-strip $a > 0$, $q_0 > q > q_\infty$ of the plane q, a is covered by the single-parameter family of integral curves, where at the same time, at each point in this region, one and only one curve passes through it. An example of curve distributions is shown in Fig. 3.

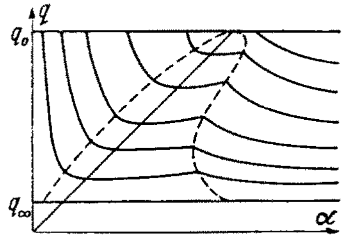

Fig. 3:

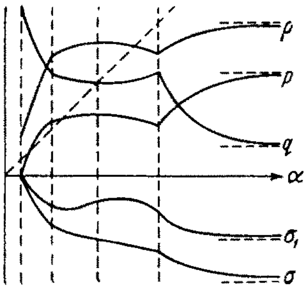

Fig. 4.

Figure 4 shows an example of ρ, p, σ, etc. distributions along α.

5. We will show that in the plastic region $\lambda > 0$ everywhere. For the case considered Formulas (1.4) give

$$\lambda = \pm \frac{1}{\sqrt{3F(p)}} \left\{ \frac{\partial}{\partial t} \left[2G \ln \frac{q}{\alpha} \mp \sqrt{3F(p)} \right] \right\} e\,[J_2 - F(p)] \times$$

$$\times\, e \left\{ \pm \frac{\partial}{\partial t} \left[2G \ln \frac{q}{\alpha} \mp \sqrt{3F(p)} \right] \right\} \qquad (5.1)$$

In this expression differentiation with respect to t is for fixed z and is equivalent to differentiation with respect to the monotonically varying boundary parameters. It is necessary to verify that, in fact, in the plastic region, i.e. for $J_2 = F(p)$, the expression

$$\pm (\partial / \partial t)\,[2G \ln (q / \alpha) \mp \sqrt{3F(p)}] \equiv \partial A / \partial t \qquad (5.2)$$

is everywhere negative. For the given boundary conditions the solution of the problem is defined by the value of the parameter which isolates q, α and by the values of α at the ends of the part of this integral curve corresponding to the boundary-value problem. If one takes the value $\alpha = \alpha_{00}$ as such a parameter for which $q = \alpha_{00}$, and takes into account that the deformation corresponding to any boundary-value problem may be considered as the succession of hydrostatic deformation leading to the attainment of the parameter α_{00}, and further deformation at constant α_{00}, then, since for hydrostatic deformation $J_2 \equiv 0$, it is not necessary to consider the dependence of A on α_{00} in evaluating $\partial A / \partial t$. The values of the parameter α at the ends of the considered part of the integral curve were worked out by monotonic change (during monotonic deformation) from their initial values, equal to α_{00} at the end of the hydrostatic stage of deformation. In the region $q > \alpha$, where $\alpha < \alpha_{00}$, these values could be obtained, apparently, only by decrease from α_{00}. In the region $q > \alpha$, where $\alpha > \alpha_{00}$, they were obtained by increase from α_{00}. Thus, one may consider that

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial \alpha}\frac{\partial \alpha}{\partial t}$$

where α_{00} is taken constant in differentiating A and α is considered as the boundary value of α corresponding to the end of the part in the integral curve considered. Carrying out the differentiation in (5.2) we obtain

$$\frac{\partial A}{\partial t} = \pm\left[\frac{q_\alpha}{q}\left(2G \pm \frac{3F'\rho_{00}\varphi'}{2q\sqrt{3F}}\right) - \frac{2G}{\alpha}\right]\frac{\partial \alpha}{\partial t} \tag{5.3}$$

Here the upper sign refers to the region $q > \alpha$. In this region the deformation after the hydrostatic stage is a rarefaction and, as just mentioned, the boundary value of α should decrease monotonically, i.e. $\partial\alpha/\partial t < 0$. Inasmuch as, in addition $q_\alpha < 0$, then in this region $\partial A/\partial t > 0$, i.e. $\lambda > 0$. In the region $q < \alpha$ there should be $\partial\alpha/\partial t > 0$, therefore for λ to be positive the following condition must obtain:

$$\frac{2G}{\alpha} + \frac{q_\alpha}{q}\left(\frac{3F'\rho_{00}\varphi'}{2q\sqrt{3F}} - 2G\right) > 0 \tag{5.4}$$

Substituting in the above the expression for q_α from (4.5) and performing certain transformations we obtain

$$\alpha > q\left\{1 + \frac{2}{3}\frac{\sqrt{3F}}{1 + F'/\sqrt{3F}}\left[\frac{3F'}{4G\sqrt{3F}} - \frac{q}{\rho_{00}\varphi'}\right]\right\} \equiv \alpha_p(q) \tag{5.5}$$

To prove (5.5) it is sufficient, apparently, to show that $\alpha_p(q) < \alpha_2(q)$ where $\alpha_2(q)$ is a function from (3.10), since the inequality (5.5) must take place only in the plastic range to the right of the boundary of (3.10). The condition $\alpha_p < \alpha_2$ is reduced to

$$\left[\frac{F'}{\sqrt{3F}} \Big/ \left(1 + \frac{F'}{\sqrt{3F}}\right)\right] < \left\{\exp\left[\frac{1}{2G}\sqrt{3F}\right] - 1\right\} \Big/ \frac{1}{2G}\sqrt{3F}$$

which is always satisfied, apparently, since the left-hand side is always smaller than and the right-hand side is always greater than unity ($F' \geqslant 0$). This fully proves the assertion that $\lambda > 0$ everywhere in the plastic region.

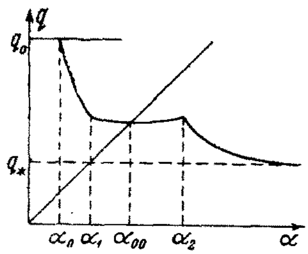

Fig. 5.

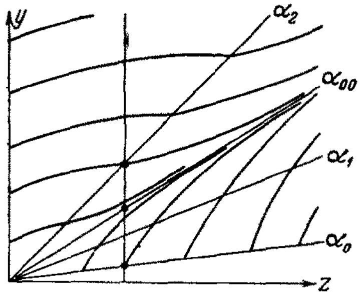

Fig. 6.

6. Let us consider the construction of boundary-value problems.

For the solution of such problems it is necessary to integrate the equation

$$y_z = q(y/z, c) = q(\alpha, c) \tag{6.1}$$

where C is a parameter for the family of integral curves on the plane q, a. The general solution of this equation is given by the quadrature

$$\ln \frac{z}{z_0} = \int_{y_0/z_0}^{y/z} \frac{d\alpha}{q(\alpha, c) - \alpha} \tag{6.2}$$

where y_0 and z_0 are arbitrary.

Let us consider the field of integral curves for Equation (6.1). We will select some integral curve from the plane q, a and will choose for the parameter C a value $a = a_{00}$ at the intersection point of the integral curve and the bisector of the coordinate angle $q = a$. Along this curve a and q will change in the region $a_0 < a < \infty$, $q_0 > q > q_*$, where a_0, q_0 correspond to the point of soil loosening, q_* is the asymptotic value of q for given a_{00}. The curve is shown in Fig. 5. Points with abscissas a_1, a_2 correspond to the boundaries between the elastic region $a_1 < a < a_2$ and the plastic regions. On the plane y, z the lines a = const are rays running from the origin of the coordinates with constant field slope. The angle $0 \leqslant z \leqslant y/a_0$ is the region where the solution (6.2) is defined.

The ray $y = a_{00}z$ is a solution of (6.2). For the lower rays, i.e. $a < a_{00}$, $q(a) > a$, while for the rays with $a > a_{00}$, $q(a) < a$. In view of the homogeneity of (6.1) all integral curves for $a < a_{00}$ can be obtained from any one of them by means of similarity transformation. It will be analogous with the curves for $a > a_{00}$. All this permits one to conclude that the origin of the coordinates contains only one particular solution $y = a_{00}z$, while all others, starting on the straight lines $a = a_0$ and

$a = \infty$, are asymptotically approaching the particular solution. The slope of the integral curves is discontinuous on the rays $a = a_1$, $a = a_2$. An example of these curves is shown in Fig. 6.

The particular solution $y = a_{00}z$ corresponds to the condition of homogeneous deformation with constant density, pressure and hydrostatic stress $\sigma_1 = \sigma = -p$. On the one hand it describes solutions of special boundary-value problems allowing such deformation, and on the other hand, since all integral curves for given a_{00} approach asymptotically this solution for $z \to \infty$, it expresses the natural property of the external solutions for the region $z_0 \leqslant z \leqslant \infty$ existing in the damping of disturbances at infinity.

Since $q(a) > q(a_{00}) = a_{00}$ for $a < a_{00}$, the density increases from the loose density at $y_0 = a_0 z_0$ to the undisturbed density $\rho'_{00} = \rho_{00}/q(a_{00}) = \rho_{00}/a_{00}$ at $z \to \infty$ on the integral curves inside the angle $a_0 < a < a_{00}$, these integral curves describe rarefaction. Thereby, the shear is elastic for the angle $a_1 < a < a_{00}$ and plastic for $a_0 < a < a_1$.

Inside the angle $a_{00} < a < \infty$ the integral curves describe the deformation resulting from the application of additional compressive boundary stresses $(\sigma < \sigma'_{00} < 0)$. Then, as was noted above, the medium elongates (angle $a_{00} < a < a_2$) during elastic shear, and only in passing to the plastic state does compressive deformation occur with compression of the medium resulting.

The end parts of the integral curves correspond to the problems of deformation in spherical layers of a finite thickness medium for given displacement of the boundaries, corresponding to the given end points on the integral curve, or under the action of stresses σ applied on the boundaries corresponding to these points. If the end of a part of the integral curve considered is on the ray $a = a_0$, this means that the soil on the inner surface of the layer has been loosened. If the curve end is located on the ray $z = 0$ this corresponds to the problem when the initial radius of the inner boundary for the medium was equal to zero.

It is interesting to note that since on this ray $a = \infty$ and p, ρ, σ etc. depend only on a for given a_{00}, this means that for all integral curves originating on the axis $z = 0$, the same σ, the same pressure p, etc correspond to the initial point. This indicates that for given a_{00}, i.e. given initial states, there must be a fully determined value of the cavity-enlarging stress $\sigma = \sigma[q_*(a_{00})] = \sigma_*$ for a cavity to be formed from a point. For $\sigma > \sigma_*$ no cavity is formed at all. For $\sigma = \sigma_*$ the cavity appears with an arbitrary radius and for $\sigma < \sigma_*$ a balanced deformation of the medium is not possible at all.

Let us draw a vertical $z = z_0$ on the plane y, z (Fig. 6) and consider the external problem for the region $z_0 \leqslant z \leqslant \infty$. The section of the vertical between the rays $a = a_{00}$ and $a = a_0$ is a set of points corresponding to the various degrees of removal of the initial stress σ'_{00} up to complete removal leading to loosening. The section between $a = a_{00}$ and $a = a_2$ corresponds to the application of additional compressive stresses $-\sigma$ for which, however, the medium outside the cavity remains everywhere in the elastic state (in shear). The section above the ray $a = a_2$ corresponds to the application of still larger compressive stresses leading to the transition of a certain layer in the medium around the cavity into the plastic state (in shear). It is important to note here that for unlimited recession upward along this vertical the compressive stress $-\sigma(z_0)$ remains bounded, tending to the above-mentioned limit $-\sigma_*$. This indicates that in the case when the cavity is widening in an infinite medium from a finite (and not zero as above) initial radius, the stress which is creating the balanced widening of the cavity is bounded from above. If $\sigma > \sigma_*$ an equilibrium is possible and a finite radius is formed:

$$\sqrt[3]{y\,(z_0)}$$

For $\sigma \to \sigma_*$ we have $y(z_0) \to \infty$, i.e. the radius of the cavity is increasing without limit.

We will show, finally, how to construct the solution in the general case when two points z_0, y_0 and z_1, y_1 are given, through which one must pass the integral curve (problem with given displacements). Clearly, these points should be given such that the inequalities $z_1 > z_0$, $y_1 > y_0$ or $z_1 < z_0$, $y_1 < y_0$ would take place.

In this case one should first choose a_{00} so that both points would fall into the region $0 \leqslant z \leqslant y/a_0$. At the same time a certain integral curve will pass through the point z_0, y_0. Should it pass above the second point z_1, y, (if $z_1 > z_0$) or below (if $z_1 < z_0$) then it is necessary to decrease a_{00} until the second point falls on the integral curve drawn through the first point. In the opposite case it is necessary to increase a_{00} and thus make the integral curve pass through both points.

The fact that with the indicated changes in a it is possible to connect both points with a solution stems from the monotonic dependence of slopes in the linear elements of the field of Equation (6.1) on the parameters a_{00}. Indeed, the smaller a_{00} the lower is the corresponding integral curve in the plane q, a, i.e. it diminishes monotonically with decrease in a_{00} for fixed a.

Therefore for fixed z and y (fixed a) the slope $q = dy/dz$ will decrease with decrease in α_{00}. This monotonic property accounts for the assertions made previously in regard to the possibility of solving the problem with arbitrary z_0, y_0 and z_1, y_1. The uniqueness of solution is established as follows. Should two integral curves corresponding to two different α_{00} pass through two different points (only one curve passes through each point for given α_{00}) then, as it is clear from geometry, the following inequalities should be satisfied:

$$q(\alpha_1; \alpha_{00}') < q(\alpha_1; \alpha_{00}''), \qquad q(\alpha_2; \alpha_{00}') > q(\alpha_2; \alpha_{00}'')$$

This is impossible since the curves $q = q(\alpha; \alpha_{00}')$ and $q = q(\alpha; \alpha_{00}'')$ cannot intersect.

7. Let us investigate the solution for elastic shear in greater detail. From the relations (3.9) and (3.10) it appears that if the elastic stresses are small compared to the shear modulus G, i.e., as is usually the case for elastic deformation, the stresses are relatively small, then $\alpha_1(q)$ and $\alpha_2(q)$ will differ but slightly from q, so that the region in the plane q, α in which the solution is described by the integral curves of Equation (3.2) will be quite narrow in the direction of the α-axis. Inside this region for $\alpha = q$ the integral curves possess horizontal tangents; therefore, everywhere in the indicated region the curves will differ but slightly from these tangents.

This shows that the elastic shear is sufficiently accurately described by the simple relation q = const in the variables q, α. But this corresponds to ρ = const, p = const. Formulas (3.1) here become

$$y_z = \frac{\rho_{00}}{\rho_1} = c, \qquad y = cz + c_1, \qquad p = -\varphi\left(\frac{\rho_{00}}{c}\right) = p_1$$

$$\sigma = -p_1 - \frac{4}{3} G \frac{c_1}{cz}, \qquad \sigma_1 = -p_1 + \frac{2}{3} G \frac{c_1}{cz} \ (c,\ c_1 = \mathrm{const}) \qquad (7.1)$$

taking into account the relation $c_1/cz \ll 1$ stemming from the narrowness of the elastic zone in the plane q, α (i.e. from the condition $|\alpha_{1,2} - q| \ll q$). But the relation (7.1) is none other than the problem solution in the ordinary theory of elasticity.

In this solution the above-established (see (3.4)) effect of density decrease in elastic shear during loading (along σ) or unloading does not now take place, i.e. it appears connected with small quantities of higher order compared with those which are important in the investigation of elastic shear. Complete solutions of boundary-value problems can be constructed joining the solution for plastic shear (4.9) not with the exact solutions of Equation (3.2) but with the simple relation q = const, sufficiently well defining the elastic shear. Also, the solution of the problem is obtained in the form of finite formulas with a number of arbitrary constants required for the solution of boundary-value problems, and it is given by the relations (4.3), (4.4), (4.11) and (7.1).

In all the considerations above we assumed that the process of volumetric deformation occurred in such a way that the pressure could be considered a well-defined function of density, i.e. processes of monotonic loading were considered in which the existence of numerous branches of unloading in dependence $p = f^{\circ}(\rho, \rho_*)$ did not occur. In considering unloading and nonmonotonic processes of loading in general, the solution of the problem becomes more difficult, although not in principle. If the loading program is given the solution of the problem will be constructed by means of consecutive considerations of subsequent monotonic components of the loading program. However, it is clear that for a sufficiently fanciful program such a consideration may represent quite an unwieldy problem. One may note also that in contrast to the monotonic processes for which one can construct a general solution in the form of finite relationships and carry out a full qualitative analysis, this cannot be done for nonmonotonic processes and the problems should be considered concretely.

We do not consider here the cylindrical problem which is midway between the plane and the spherical. It can be treated analogously, and it appears that the qualitative effects of this problem will be midway between those of the two cases considered.

BIBLIOGRAPHY

1. Grigorian, S.S., Ob obshchikh uravneniiakh dinamiki gruntov (On the general equations of soil dynamics). *Dokl. Akad. Nauk SSSR* Vol.124, No. 2, 1959.

2. Grigorian, S.S., Ob osnovnykh predstavleniiakh dinamiki gruntov (On the basic concepts of soil dynamics). *PMM* Vol. 24, No. 6, 1960.

3. Alekseenko, V.D., Grigorian, S.S., Novgorodov, A.F. and Rykov, G.V., Nekotorye eksperimental'nye issledovaniia po dinamike miagkikh gruntov (Some experimental investigations of soft-soil dynamics). *Dokl. Akad. Nauk SSSR* Vol. 133, No. 6, 1960.

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